Signals and Systems

Signals
Systems
The Fourier Transform
Properties of the Fourier Transform
Transfer Function
Circular Symmetry and the Hankel Transform
Sampling

• signals and systems
  - two fundamental concepts for modeling medical imaging systems
  - useful for modeling how physical processes (signals) change
    how systems create new signals (i.e., image)

• signals
  - mathematical functions of a variety of physical processes
    ① continuous: distri. of x-ray attenuation \( f(x, y) \) in a x-sec. w/\( i \) the human body in CT
    ② discrete: representation of \( f(x, y) \) in a computer, \( f_d(m, n) \)
    ③ mixed signals: sinogram in CT, \( g(l, \theta_k) \) where \( k = 1, 2, \ldots \)

• systems
  - respond to signals by producing new signals
  - continuous-to-continuous and continuous-to-discrete systems

Note that the challenge in medical imaging is to make the output signal (coming from the imaging system) a faithful representation of the input signal (coming from the patient)!
Signals

For a function or a signal $f(x, y)$, you may plot it as a function of the two indep. variables, or you may display it, by assigning an intensity or brightness proportional to its value at $(x, y)$.

- various signals:
  - point impulse
  - comb and sampling functions
  - line impulse
  - rect and sinc functions
  - exponential and sinusoidal signals
Point impulse

Useful to model the concept of a "point source", which is used in the characterization of imaging system resolution

- 1-D point impulse (the concept of a point source in 1-D)

  - delta function, Dirac function, impulse function

  \[ \delta(x) = 0, \quad x \neq 0 \]

  \[ \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \]

  - not a function in usual sense, but instead acting on other signals through integration
  - modeling the property of a point source by having infinitesimal width and unit area

  \[ \Rightarrow \text{If } f(x) = 1, \quad \int_{-\infty}^{\infty} \delta(x)dx = 1 \]

- 2-D point impulse

  - 2-D delta function, 2-D Dirac function, 2-D impulse function

  \[ \delta(x, y) = 0, \quad (x, y) \neq (0, 0) \]

  \[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x, y)dxdy = f(0, 0) \]

  Note that the point impulse "picks off" the value of \( f(x, y) \) at the location \((0, 0)\) by multiplying \( f(x, y) \) with the point impulse followed by integration over all space.

  - modeling the property of a point source by having infinitesimal width and unit volume
• properties of the point impulse

shifting \[ f(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x-\xi, y-\eta)dx\,dy \]

scaling \[ \delta(ax, by) = \frac{1}{|ab|}\delta(x, y) \]

even function \[ \delta(-x,-y) = \delta(x, y) \]
■ Line impulse

- a set of points: \[ L(\ell, \theta) = \{(x, y) \mid x \cos \theta + y \sin \theta = \ell\} \]
- line impulse: \[ \delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - \ell) \]

■ Comb and sampling functions

In medical imaging modalities, we need to "pick off" values on a grid or matrix of points (called sampling) by using shifted point impulses

=> this matrix of values represents the medical image

- comb function: \[ \text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \quad \text{also called shah function} \]

\[ \text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) \]

- sampling: a process picking off values not just at a single point, but on a grid or matrix of points \( \rightarrow \) digital image

- sampling function: \[ \delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \]

\[ \delta_s(x, y; \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \text{comb} \left( \frac{x}{\Delta x}, \frac{y}{\Delta y} \right) \]

- a sequence of point impulses located at points \((m\Delta x, n\Delta y)\) of the plane
Rect and sinc functions

- useful in understanding the influence of the finite width of pixels in portraying an originally continuous signal as a discrete image

- rectangular function or normalized *boxcar* function

\[
\text{rect}(x, y) = \begin{cases} 
1, & \text{for } |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\
0, & \text{for } |x| > \frac{1}{2} \text{ or } |y| > \frac{1}{2} 
\end{cases}
\]

\[
\text{rect}(x, y) = \text{rect}(x)\text{rect}(y)
\]

where

\[
\text{rect}(x) = \begin{cases} 
1, & \text{for } |x| < \frac{1}{2} \\
0, & \text{for } |x| > \frac{1}{2} 
\end{cases}
\]

- a finite energy signal with unit total energy
- modeling signal concentration over a unit square centered around point (0, 0)
- e.g., to select that part of signal \( f(x, y) \) centered at a point \((\xi, \eta)\) of the plane with width \(X\) and height \(Y\)

\[
f(x, y)\text{rect}\left(\frac{x - \xi}{X}, \frac{y - \eta}{Y}\right)
\]
sinc function

\[
sinc(x, y) = \begin{cases} 
1, & \text{for } x = y = 0 \\
\frac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}, & \text{otherwise}
\end{cases}
\]

\[
sinc(x, y) = sinc(x)sinc(y)
\]

where \( sinc(x) = \frac{\sin(\pi x)}{\pi x} \)

- a finite energy signal with unit total energy
- consisting of a main lobe and several side lobes, which eventually diminishing to zero
- Exponential and sinusoidal signals

- complex exponential signal

\[ e(x, y) = e^{j2\pi(u_0x + v_0y)} \]

- decomposed into real and imaginary parts using sinusoidal signals

\[ e(x, y) = e^{j2\pi(u_0x + v_0y)} = \cos[2\pi(u_0x + v_0y)] + j\sin[2\pi(u_0x + v_0y)] \]

\[ = c(x, y) + js(x, y) \]

where \[ s(x, y) = \sin[2\pi(u_0x + v_0y)] = \frac{1}{2j}e^{j2\pi(u_0x + v_0y)} - \frac{1}{2j}e^{-j2\pi(u_0x + v_0y)} \]

\[ c(x, y) = \cos[2\pi(u_0x + v_0y)] = \frac{1}{2}e^{j2\pi(u_0x + v_0y)} + \frac{1}{2}e^{-j2\pi(u_0x + v_0y)} \]

\[ u_0, v_0: \text{fundamental frequencies} \] affecting the oscillating behavior of sinusoidal signals in the corresponding directions

small values \( \rightarrow \) slow oscillation, large values \( \rightarrow \) fast oscillation

dimension = \([x]^{-1} \& [y]^{-1}\)
Separable signals

A signal \( f(x, y) \) is separable if \( f(x, y) = f_1(x)f_2(y) \).

- e.g. point impulse: \( \delta(x, y) = \delta(x)\delta(y) \)
- rect function
- sinc function

Periodic signals

A signal \( f(x, y) \) is periodic if \( f(x, y) = f(x + X, y) = f(x, y + Y) \).

where \( X \) and \( Y \) are the signal periods.

- e.g. sampling function \( \delta_s(x, y; \Delta x, \Delta y) \) with periods \( X = \Delta x \) and \( Y = \Delta y \).
- exponential sinusoidal signals with periods \( X = 1/u_0 \) and \( Y = 1/v_0 \).
**Systems**

\[ g(x, y) = S[f(x, y)] \]

- \( S[\cdot] \) = a response of a system or a transformation of an input signal
- predicting the output of an imaging system
  - if knowing the input \( f(x, y) \) and the characteristics of the system \( S[\cdot] \)

- **Linear systems**

  For any collections \( \{f_k(x, y), k = 1, 2, \ldots, K\} \) of input signals and \( \{w_k, k = 1, 2, \ldots, K\} \) of weights;

  \[
  S\left[ \sum_{k=1}^{K} w_k f_k (x, y) \right] = \sum_{k=1}^{K} w_k S[f_k (x, y)] \implies "linear system"
  \]

"Linear-system assumption means that the image of an ensemble of signals is identical to the sum of separate images of each signal."
Impulse response

Consider the output of a system \( S[\cdot] \) to a point impulse located at \((\xi, \eta)\): \( \delta_{\xi \eta}(x, y) = \delta(x - \xi, y - \eta) \):

\[ h(x, y; \xi, \eta) = S[\delta_{\xi \eta}(x, y)] \]

point spread function (PSF) or impulse response function (IRF)

Assuming a linear system:

\[
g(x, y) = S[f(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S[f(\xi, \eta) \delta_{\xi \eta}(x, y)] d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) S[\delta_{\xi \eta}(x, y)] d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x, y; \xi, \eta) d\xi d\eta
\]

Therefore, for any linear system with PSF \( h(x, y; \xi, \eta) \), the input-output equation is

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x, y; \xi, \eta) d\xi d\eta \quad \text{"superposition integral"}
\]

\[ \rightarrow \text{PSF uniquely characterizes a linear system} \]

\[ \rightarrow \text{only need to know the PSF} h \text{ to calculate the output} g \text{ for a given input} f \]

\[ \rightarrow \text{not practical} \text{ to get the PSF} h \text{ for every} \ (x, y; \xi, \eta) \]

Shift invariance

: if an arbitrary translation of the input results in an identical translation in the output

\[
g(x - x_0, y - y_0) = S[f_{x_0 y_0}(x, y)]
\]

where \( f_{x_0 y_0}(x, y) = f(x - x_0, y - y_0) \)

the response to a translated input = that to the actual input translated by the same amount

\[ \Rightarrow \text{shift-invariant system} \]
If a system is linear and shift-invariant (LSI), and \( h(x, y) = S[\delta(x, y)] \);

\[
S[\delta(x, y)] = h(x - \xi, y - \eta)
\]

"2-D PSF"

→ the PSF is the same throughout the field-of-view (FOV) of a shift-invariant system

→ need a measurement of PSF at only one position

For an LSI system:

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)h(x - \xi, y - \eta)d\xi d\eta \quad \text{convolution integral}
\]

or \( g(x, y) = h(x, y) \otimes f(x, y) \quad \text{convolution equation} \)

**Connections of LSI systems**

1. cascade or serial connections

\[
g(x, y) = h_2(x, y) \otimes [h_1(x, y) \otimes f(x, y)]
\]

\[
= h_1(x, y) \otimes [h_2(x, y) \otimes f(x, y)] \quad \text{associativity}
\]

\[
= [h_1(x, y) \otimes h_2(x, y)] \otimes f(x, y)
\]

\[
h_1(x, y) \otimes h_2(x, y) = h_2(x, y) \otimes h_1(x, y) \quad \text{commutativity}
\]

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)h(x - \xi, y - \eta)d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta)d\xi d\eta
\]
parallel connections

\[ g(x, y) = h_1(x, y) \otimes f(x, y) + h_2(x, y) \otimes f(x, y) \]
\[ = [h_1(x, y) + h_2(x, y)] \otimes f(x, y) \]

**Separable systems**

\[ h(x, y) = h_1(x)h_2(y) \]

- often faster (and easier) to execute two consecutive 1-D operations than a single 2-D operation
- 2-D convolution integral calculation procedure using two simpler 1-D convolution integrals

1. \[ w(x, y) = \int_{-\infty}^{\infty} f(\xi, y)h_1(x - \xi)\,d\xi \quad \text{for every } y \quad \text{fixing } y \text{ and convolving with } h_1(x) \]

2. \[ g(x, y) = \int_{-\infty}^{\infty} w(x, \eta)h_2(y - \eta)d\eta \quad \text{for every } x \quad \text{fixing } x \text{ and convolving with } h_2(y) \]

\[ \int_{-\infty}^{\infty} w(x, \eta)h_2(y - \eta)d\eta = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi, \eta)h_1(x - \xi)d\xi\right]h_2(y - \eta)d\eta \]
\[ \Rightarrow \]
\[ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)h_1(x - \xi)h_2(y - \eta)d\xi d\eta \]
\[ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)h(x - \xi, y - \eta)d\xi d\eta \]
\[ = h(x, y) \otimes f(x, y) = g(x, y) \]
• e.g., Convolution of an image with a 2-D Gaussian PSF
  \[ h(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2 + y^2)}{2\sigma^2}} \] with \( \sigma = 4 \)

by using two 1-D steps in cascade:
  \[ h_1(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \] and
  \[ h_2(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}} \]

Stable systems
  : **stable** if small inputs lead to outputs that do not diverge

• a **bounded-input bounded-output** (BIBO) stable system:
  there exists a finite \( B' \) (bounded-output) for some finite \( B \) (bounded-input)

  \[ |g(x, y)| = |h(x, y) \otimes f(x, y)| \leq B' < \infty \quad \text{for} \quad |f(x, y)| \leq B < \infty, \quad \text{for every} \quad (x, y) \]

• an LSI system is a BIBO stable system if and only if its PSF is absolutely integrable;

  \[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(x, y)| \, dx \, dy < \infty \]
The Fourier Transform

- alternative (& equivalent) way relating the output of LSI syst. to its input instead of the convolution
- decomposing a signal into point impulses ⇒ convolution
- decomposing a signal in terms of complex exponential signals ⇒ Fourier transform

- 2-D Fourier transform of \( f(x, y) \): \( F(u, v) = \mathcal{F}_{2D}(f)(u, v) \)

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy
\]

- \( u \) & \( v \) = \( x \) & \( y \) spatial frequencies
- complex-valued signals

\[
F(u, v) = F_R(u, v) + jF_I(u, v) = |F(u, v)|e^{j\angle F(u, v)}
\]

- magnitude spectrum \( |F(u, v)| = \sqrt{F_R^2(u, v) + F_I^2(u, v)} \)
- power spectrum \( |F(u, v)|^2 \)
- phase spectrum \( \angle F(u, v) = \tan^{-1}\left(\frac{F_I(u, v)}{F_R(u, v)}\right) \)

- spectral decomposition of functions or sequences in terms of their harmonic content
- providing information on the sinusoidal composition of a signal \( f(x, y) \) at different frequencies

- 1-D Fourier transform: \( F(u) = \mathcal{F}_{1D}(f) (u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi xu} \, dx \)

What is the physical meaning of the magnitude and phase in an image?

- 2-D inverse Fourier transform: \( f(x, y) = \mathcal{F}_{2D}^{-1}(F)(x, y) \)

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} \, dx \, dy
\]

- 1-D inverse Fourier transform: \( f(x) = \mathcal{F}_{1D}^{-1}(F)(x) = \int_{-\infty}^{\infty} F(u)e^{j2\pi xu} \, du \)
Ex) \[ \mathcal{F}\{\delta(x,y)\} = ? \]
\[ \mathcal{F}\{e^{j2\pi(u_0x+v_0y)}\} = ? \]
\[ \mathcal{F}\{\text{rect}(x)\} = ? \]
\[ \mathcal{F}\{\text{sinc}(u)\} = ? \]

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<th>Fourier transform</th>
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<td>1</td>
<td>[ \delta(u,v) ]</td>
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<tr>
<td>[ \delta(x,y) ]</td>
<td>[ 1 ]</td>
</tr>
<tr>
<td>[ \delta(x-x_0,y-y_0) ]</td>
<td>[ e^{-j2\pi(u_0x+v_0y)} ]</td>
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<tr>
<td>[ \delta_\Delta(x,y;\Delta x,\Delta y) ]</td>
<td>[ \text{comb}(u\Delta x,v\Delta y) ]</td>
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<tr>
<td>[ e^{j2\pi(u_0x+v_0y)} ]</td>
<td>[ \delta(u-u_0,v-v_0) ]</td>
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<tr>
<td>[ \sin[2\pi(u_0x+v_0y)] ]</td>
<td>[ \frac{1}{2j} \left[ \delta(u-u_0,v-v_0) - \delta(u+u_0,v+v_0) \right] ]</td>
</tr>
<tr>
<td>[ \cos[2\pi(u_0x+v_0y)] ]</td>
<td>[ \frac{1}{2} \left[ \delta(u-u_0,v-v_0) + \delta(u+u_0,v+v_0) \right] ]</td>
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<tr>
<td>[ \text{rect}(x,y) ]</td>
<td>[ \text{sinc}(u,v) ]</td>
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<td>[ \text{comb}(x,y) ]</td>
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</tr>
<tr>
<td>[ e^{-\pi(x^2+y^2)} ]</td>
<td>[ e^{-\pi(u^2+v^2)} ]</td>
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</table>
Properties of the Fourier Transform

- **Linearity**

\[ \mathcal{F}_2D(a_1f + a_2g)(u,v) = a_1 F(u,v) + a_2 G(u,v) \]

- **Translation**

\[ \mathcal{F}_2D(f_{x_0,y_0})(u,v) = F(u,v)e^{-j2\pi(u x_0 + v y_0)} \]

where \( f_{x_0,y_0}(x, y) = f(x - x_0, y - y_0) \)

\[ |\mathcal{F}_2D(f_{x_0,y_0})(u,v)| = |F(u,v)| \]

\[ \angle \mathcal{F}_2D(f_{x_0,y_0})(u,v) = \angle F(u,v) - 2\pi(u x_0 + v y_0) \]

- Translating a signal does not affecting its magnitude spectrum
  but subtracts a constant phase of \( 2\pi(u x_0 + v y_0) \) at each frequency \( (u, v) \)

- **Conjugation and conjugate symmetry**

\[ \mathcal{F}_2D(f^*)(u,v) = F^*(-u,-v) \]

\[ F(u,v) = F^*(-u,-v) \]

\[ F_r(u,v) = F_r(-u,-v), \quad |F(u,v)| = |F(-u,-v)| \]

\[ F_i(u,v) = -F_i(-u,-v), \quad \angle F(u,v) = -\angle F(-u,-v) \]

**What is the physical meaning of the complex conjugate?**

- **Scaling**

\[ f_{ab}(x, y) = f(ax, by) \quad \iff \quad \mathcal{F}_2D(f_{ab})(u,v) = \frac{1}{|ab|} F \left( \frac{u}{a}, \frac{v}{b} \right) \]
■ Rotation

\[ f_\theta(x, y) = f(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \]
\[ \Leftrightarrow \mathcal{Z}_{2D}(f_\theta)(u, v) = F(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) \]

■ Convolution

\[ \mathcal{Z}_{2D}(f \otimes g)(u, v) = F(u, v)G(u, v) \quad \text{convolution theorem} \]

■ Product

\[ \mathcal{Z}_{2D}(fg)(u, v) = F(u, v) \otimes G(u, v) \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi, \eta)F(u - \xi, v - \eta)d\xi d\eta \]

■ Separable product

\[ f(x, y) = f_1(x)f_2(y) \quad \Leftrightarrow \quad \mathcal{Z}_{2D}(f)(u, v) = F_1(u)F_2(v) \]

where
\[ F_1(u) = \mathcal{Z}_{1D}(f_1)(u) \]
\[ F_2(v) = \mathcal{Z}_{1D}(f_2)(v) \]

■ Parseval's theorem

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u, v)|^2 du dv \]

• the total energy of a signal \( f(x, y) \) in the spatial domain = its total energy in the frequency domain
• the Fourier transformation as well as its inverse are energy-preserving ("unit gain") transformations
Separability

- 2-D Fourier transformation procedure using two simpler 1-D Fourier transforms

\[ r(u, y) = \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi xu} \, dx \quad \text{for every } y \]

\[ F(u, v) = \int_{-\infty}^{\infty} r(u, y) e^{-j2\pi yv} \, dy \quad \text{for every } u \]

\[
\int_{-\infty}^{\infty} r(u, y) e^{-j2\pi yv} \, dy = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi xu} \, dx \right] e^{-j2\pi yv} \, dy
\]

\[
\Rightarrow \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi (ux + yv)} \, dx \, dy
\]

\[
= F(u, v)
\]
Properties of the Fourier transform

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<td>$\angle F(u,v) = 0$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$f(x,y) \otimes g(x,y)$</td>
<td>$F(u,v)G(u,v)$</td>
</tr>
<tr>
<td>Product</td>
<td>$f(x,y)g(x,y)$</td>
<td>$F(u,v)\otimes G(u,v)$</td>
</tr>
<tr>
<td>Separable product</td>
<td>$f(x)g(y)$</td>
<td>$F(u)G(v)$</td>
</tr>
<tr>
<td>Parseval's theorem</td>
<td>$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}</td>
<td>f(x,y)</td>
</tr>
</tbody>
</table>
Transfer Function

Consider responses to complex exponential signals $e^{j2\pi(u x + vy)}$ for an LSI system:

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi(u \xi + v \eta)} h(x - \xi, y - \eta) d\xi d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) e^{j2\pi(u(x - \xi) + vy(y - \eta))} d\xi d\eta$$

$$= \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) e^{-j2\pi(u \xi + v \eta)} d\xi d\eta \right] e^{j2\pi(u x + vy)}$$

$$= H(u, v) e^{j2\pi(u x + vy)}$$

$$\Rightarrow \text{output} = H(u, v) \times \text{input}$$

where $$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) e^{-j2\pi(u \xi + v \eta)} d\xi d\eta$$

the Fourier transformation of the PSF $h(x, y)$

the transfer function of the LSI system

the frequency response or the optical transfer function (OTF) of system uniquely characterizing an LSI system because the PSF is uniquely determined by

$$h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) e^{j2\pi(u x + vy)} dudv$$
If an input $f(x, y)$ to an LSI system $S[\cdot]$ with transfer function $H(u, v)$ produces an output $g(x, y)$

$$G(u, v) = H(u, v)F(u, v)$$

- low spatial frequencies: signals varying slowly across the image
- high spatial frequencies: signals varying locally (e.g., at the edges of structures within an image)

- ideal low-pass filter with cutoff frequency $c$: 
  $$H(u, v) = \begin{cases} 
  1, & \text{for } \sqrt{u^2 + v^2} \leq c \\
  0, & \text{for } \sqrt{u^2 + v^2} > c 
  \end{cases}$$

yielding 

$$G(u, v) = \begin{cases} 
  F(u, v), & \text{for } \sqrt{u^2 + v^2} \leq c \\
  0, & \text{for } \sqrt{u^2 + v^2} > c 
  \end{cases}$$

signal smoothing

Note that $c_1 > c_2$

Note that, usually, the numerical implementation of the convolution equation is not done in the space domain, but in the frequency domain. The main reason is the existence of an efficient algorithm (the fast Fourier transform, or FFT) that allows a fast computer implementation.
Circular Symmetry and the Hankel Transform

A 2-D signal \( f(x, y) \) is circularly symmetric if \( f_\theta(x, y) = f(x, y) \) for every \( \theta \), where \( f_\theta(x, y) = f(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \)

\[ \Rightarrow \mathcal{F}_{2D}(f_\theta(u, v)) = F(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) \text{ is also circularly symmetric} \]

\[ \Rightarrow \text{giving rise to} \]

- \( f(x, y) \): even in both \( x \) and \( y \)
- \( F(u, v) \): even in both \( u \) and \( v \)
- \( |F(u, v)| = F(u, v) \) & \( \angle F(u, v) = 0 \)

Alternatively

- \( f(x, y) = f(r) \Leftrightarrow F(u, v) = F(\rho) \)

where \( r = \sqrt{x^2 + y^2} \) & \( \rho = \sqrt{u^2 + v^2} \)

- \( F(\rho) = 2\pi \int_0^\infty f(r) J_0(2\pi \rho r) r dr \) Hankel transform

or \( F(\rho) = \mathcal{H}\{f(r)\} \)

where \( J_0(r) = \) the zero-order Bessel function of the first kind

- \( n \text{-order Bessel function of the first kind} \)

\[ J_n(r) = \frac{1}{\pi} \int_0^\pi \cos(nr - r \sin \phi) d\phi, \quad n = 0, 1, 2, \ldots \]

\[ \Rightarrow J_0(r) = \frac{1}{\pi} \int_0^\pi \cos(r \sin \phi) d\phi \]

- inverse Hankel transform: \( f(r) = 2\pi \int_0^\infty F(\rho) J_0(2\pi \rho r) \rho d\rho \)
• The Fourier transform can be found using the Hankel transform if a 2-D signal is circularly symmetric

  e.g. Gaussian function \( e^{-\pi(x^2+y^2)} \)
  A good model for the blurring inherent in medical imaging systems

  \[
  H\left\{e^{-\pi x^2}\right\} = e^{-\pi r^2} ; \text{ note that } \left\{\frac{e^{-\pi(x^2+y^2)}}{2}\right\} = e^{-\pi(a^2+rv^2)}
  \]

  e.g. Unit disk \( f(r) = \text{rect}(r) \)

  \[
  H\{\text{rect}(r)\} = \frac{J_1(\pi\rho)}{2\rho} = jink(\rho)
  \]

  where \( J_1 \) = the first-order Bessel function of the first kind

  ![Graph of jinc(x) and jinc(x)]

  \( jinc(0) = \frac{\pi}{4} \)

  \( jinc(1.2197) = 0 \)

  \( jinc(2.2331) = 0 \)

  \( jinc(3.2383) = 0 \)

  half the peak value of jinc function at \( x = 0.70576 \)

  - scaling: \( H\{f(ar)\} = \frac{1}{a^2} F(\rho/a) \) recognizing \( a = b \) for circularly symmetry
<table>
<thead>
<tr>
<th>Signal</th>
<th>Hankel transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-\pi r^2}$</td>
<td>$e^{-\pi r^2}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{\delta(\rho)}{\pi \rho} = \delta(u, v)$</td>
</tr>
<tr>
<td>$\delta(r-a)$</td>
<td>$2 \pi a J_0(2\pi au)$</td>
</tr>
<tr>
<td>$\text{rect}(r)$</td>
<td>$\frac{J_1(\pi \rho)}{2 \rho}$</td>
</tr>
<tr>
<td>$\text{sinc}(r)$</td>
<td>$\frac{2 \text{rect}(\rho)}{\pi \sqrt{1-4\rho^2}}$</td>
</tr>
<tr>
<td>$\frac{1}{r}$</td>
<td>$\frac{1}{\rho}$</td>
</tr>
</tbody>
</table>
Sampling

- *discretization* or *sampling*
  - transforming continuous signals into collections of numbers
  - retaining representative signal values and discard the rest
  - rectangular sampling scheme
    \[ f_d(m, n) = f(m\Delta x, n\Delta y), \text{ for } m, n = 0, 1, 2, \ldots \]
    where \( \Delta x, \Delta y = \text{sampling periods} \rightarrow \frac{1}{\Delta x}, \frac{1}{\Delta y} = \text{sampling frequencies} \)

*Given a 2-D continuous signal \( f(x, y) \), what are the maximum possible values of \( \Delta x \) and \( \Delta y \) such that \( f(x, y) \) can be reconstructed from the 2-D discrete signal \( f_d(m, n) \)?
• **aliasing**

- a type of signal corruption when sampling a continuous signal with too few samples
- higher frequencies "take the alias of" lower frequencies
- high-frequency artifact due to *undersampled* signals
- the overlap of high-frequency spectra in the aliased Fourier transform
  artificially boots high-frequency content
- appearing as new patterns where none should exist

![Sampling signal model](image)

**Sampling signal model**

Consider multiplying a continuous signal $f(x, y)$ by the sampling functions:

$$f_s(x, y) = f(x, y)\delta_s(x, y; \Delta x, \Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(x, y)\delta(x - m\Delta x, y - n\Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_s(m, n)\delta(x - m\Delta x, y - n\Delta y)$$
Taking a Fourier transformation using $\mathcal{F}_{2D}(\delta, (x, y; \Delta x, \Delta y)) = \text{comb}(u \Delta x, v \Delta y)$:

$$F_x(u, v) = F(u, v) \otimes \text{comb}(u \Delta x, v \Delta y)$$

$$= F(u, v) \otimes \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u \Delta x - m \Delta x, v \Delta y - n)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u, v) \otimes \delta(u - m \Delta x, v - n \Delta y)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta) \delta(u - m \Delta x - \xi, v - n \Delta y - \eta) d\xi d\eta$$

$$= \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - m \Delta x, v - n \Delta y)$$

$F(u, v) = \text{the spectrum of a band-limited continuous signal } f(x, y) \text{ with cutoff frequencies } U \text{ & } V$

Sampling periods $\Delta x < 1/2U$ & $\Delta y < 1/2V$ or sampling frequencies $1/\Delta x > 2U$ & $1/\Delta y > 1/2V$
Sampling periods $\Delta x > 1/2U$ & $\Delta y > 1/2V$ or sampling frequencies $1/\Delta x < 2U$ & $1/\Delta y < 2V$

- Nyquist sampling theorem

  - **band-limited** signal
    - the spectrum of $f(x, y)$ should be zero outside a rectangle in frequency space
      so that the spectra in $F_s(u, v)$ do not overlap
    - if $\Delta x \leq \frac{1}{2U}$ and $\Delta y \leq \frac{1}{2V}$, $F(u, v)$ can be reconstructed from $F_s(u, v)$
      
      in other words, $f(x, y)$ can be reconstructed from $f_s(x, y)$ [or from samples $f_d(m, n)$]
    - if $\Delta x > \frac{1}{2U}$ and $\Delta y > \frac{1}{2V}$, there is overlap of "high" frequencies of $F(u, v)$ in $F_s(u, v)$
      
      $\rightarrow$ producing aliasing

  - Nyquist sampling theorem

A 2-D continuous band-limited signal $f(x, y)$, with cutoff frequencies $U$ and $V$, can be uniquely determined from its samples $f_d(m, n) = f(m\Delta x, n\Delta y)$, if and only the sampling periods $\Delta x$ and $\Delta y$ satisfy

$$\Delta x \leq \frac{1}{2U} \quad \text{and} \quad \Delta y \leq \frac{1}{2V}.$$  

- Nyquist sampling periods

  - maximum allowed values for $\Delta x$ and $\Delta y$ are $(\Delta x)_{\text{max}} = \frac{1}{2U}$ and $(\Delta y)_{\text{max}} = \frac{1}{2V}$
Anti-aliasing filters

- an inherent trade-off in medical imaging systems
  - the number of samples (acquired by a detector) vs. image quality
  - a large number of samples → highest image resolution but expensive or time-consuming
  - reducing the number of samples → may lead to "aliasing"

- alternatives
  - filtering the continuous signal using a low-pass filter, then sampling with fewer samples
    ⇒ blurring rather than aliasing artifacts (usually preferable)
    → anti-aliasing filter applied before sampling

- most imaging detectors are integrators rather than point samplers
  → do not sample a signal at an array of points
  → but, integrate the continuous signal locally
  → not usually accurate modeling in sampling such that \( f_s(x,y) = f(x,y)\delta_s(x,y;\Delta x,\Delta y) \)
  → local integration is modeled by convolving \( f(x,y) \) with the PSF \( h(x,y) \) of the detector

\[
\begin{align*}
    f_d(m,n) &= f_s(m\Delta x,n\Delta y) \\
    f_s(x,y) &= [h(x,y) \otimes f(x,y)]\delta_s(x,y;\Delta x,\Delta y)
\end{align*}
\]