1 Mathematical Modeling, Numerical Methods, and Problem Solving

A Simple Mathematical Model
Conservation Laws in Engineering and Science
Numerical Methods Covered in This Course

\[ \frac{dy}{dt} = g - \frac{c_d}{m} v^2 \]

where
- \( v \) = vertical velocity (m/s)
- \( t \) = time (s)
- \( g \) = the acceleration due to gravity (\( \cong 9.81 \text{ m/s}^2 \))
- \( c_d \) = a second-order drag coefficient (kg/m)
- \( m \) = the jumper’s mass (kg)
A Simple Mathematical Model

- Mathematical model
  
  • a formulation expressing the essential features of a physical system in mathematical forms
  • dependent variable = \( f(\text{independent variables, parameters, forcing functions}) \)
    - dependent variable = the behavior or state of the system
    - independent variables = dimensions along which the system's behavior is being determined
      e.g., space, time
    - parameters = system's properties or composition
    - forcing functions = external influences

From Newton's second law,

\[
\frac{dv}{dt} = \frac{F}{m} = \frac{F_D + F_U}{m} = \frac{mg - c_d v^2}{m}
\]

where

\( F_D \) = downward gravity

\( F_U \) = upward air resistance

\[
:: \quad \frac{dv}{dt} = g - \frac{c_d}{m} v^2
\]

If \( v = 0 \) at \( t = 0 \), the analytical solution is

\[
v(t) = \sqrt{\frac{gm}{c_d}} \tanh \left( \sqrt{\frac{g c_d}{m}} t \right)
\]

Refer to the bottom supplement!
There are many mathematical models that cannot be solved exactly!

Therefore, a **numerical solution** that **approximates** the exact solution is needed.

### Numerical methods

- reformulating the mathematical problem
- solving with arithmetic operations

Note that

\[
\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}
\]

Using a **finite (divided) difference** approximation of the derivative,

\[
\frac{dv}{dt}\approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v^2(t_i)
\]

Rearranging,

\[
v_{i+1} = v_i + \frac{dv}{dt} \Delta t = v_i + \tan \theta \times \Delta t = v_i + \Delta v
\]

New value = old value + slope × step size

→ **Euler's method**
How can we minimize the discrepancy between the exact and numerical solutions?

- using a smaller step size
  → calculation time increases → impractical
  → needing the aid of the computer

![Graph showing exact and numerical solutions](image)
Conservation Laws in Engineering and Science

- Transient (or time-variable) analysis
  - predicting changes with respect to time
  - change = increases – decreases

\[
\frac{dv(t)}{dt} = g - \frac{c_d}{m} v^2(t) \rightarrow v(t) = \sqrt{\frac{gm}{c_d}} \tanh \left( \sqrt{\frac{gc_d}{m}} t \right)
\]

- Steady-state analysis
  - change = 0 = increases – decreases

\[
\frac{dv(t)}{dt} = 0 = g - \frac{c_d}{m} v^2(t) \rightarrow
\]
<table>
<thead>
<tr>
<th>Field</th>
<th>Device</th>
<th>Organizing Principle</th>
<th>Mathematical Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical engineering</td>
<td>Reactors</td>
<td>Conservation of mass</td>
<td>Mass balance:</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Over a unit of time period</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑m = input - outputs</td>
</tr>
<tr>
<td>Civil engineering</td>
<td>Structure</td>
<td>Conservation of momentum</td>
<td>Force balances:</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>At each node</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ horizontal forces (Fh) = 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ vertical forces (Fv) = 0</td>
</tr>
<tr>
<td>Mechanical</td>
<td>Machine</td>
<td>Conservation of momentum</td>
<td>Force balances:</td>
</tr>
<tr>
<td>engineering</td>
<td></td>
<td></td>
<td>Upward force</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ x = 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Downward force</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ -x = 0</td>
</tr>
<tr>
<td>Electrical</td>
<td>Circuit</td>
<td>Conservation of charge</td>
<td>Current balance:</td>
</tr>
<tr>
<td>engineering</td>
<td></td>
<td></td>
<td>For each node</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ current (i) = 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Conservation of energy</td>
<td>Voltage balance:</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Around each loop</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ emf = ∑ voltage drop for resistors = 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>∑ ξ + ∑ Δε = 0</td>
</tr>
</tbody>
</table>
Numerical Methods

(a) **Part 2: Roots**

Solve \( f(x) = 0 \) for \( x \)

(b) **Part 3: Linear algebraic equations**

Given the \( a \)’s and the \( b \)’s, solve for the \( x \)’s

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*}
\]

(c) **Part 4: Curve fitting**

Regression

(d) **Part 5: Integration**

\[
I = \int_a^b f(x)dx
\]

Find the area under the curve.

(e) **Part 6: Differential equations**

Given

\[
\frac{dy}{dt} = \frac{\Delta y}{\Delta t} = f(t, y)
\]

Solve for \( y \) as a function of \( t \)

\[
y_{i+1} = y_i + f(t_i, y_i)\Delta t
\]
Find the solution of \( \frac{dv(t)}{dt} = g - \frac{c_s}{m} v^2(t) \).

Try by yourself on this page. The reference solution will be available at [http://bml.pusan.ac.kr](http://bml.pusan.ac.kr) before the very next class.

Sol.)
Let us consider the free-falling velocity problem with a first-order drag coefficient, which is more general.

$$\frac{dv(t)}{dt} = g - \frac{c_d}{m} v(t); \text{first-order linear differential equation}$$

Typical first-order linear differential equation has a form of \( y' + p(x)y = r(x) \),

and the solution is that \( y(x) = e^{-\int p(x)dx} e^{\int r(x)dx} \) where \( h = \int p(x)dx \).

Therefore, we have a solution for the free-falling velocity; \textbf{Try it !!}