Cascaded-systems analyses and the detective quantum efficiency of single-Z x-ray detectors including photoelectric, coherent and incoherent interactions

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Cascaded-systems analyses and the detective quantum efficiency of single-Z x-ray detectors including photoelectric, coherent and incoherent interactions

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Purpose: Theoretical models of the detective quantum efficiency (DQE) of x-ray detectors are an important step in new detector development by providing an understanding of performance limitations and benchmarks. Previous cascaded-systems analysis (CSA) models accounted for photoelectric interactions only. This paper describes an extension of the CSA approach to incorporate coherent and incoherent interactions, important for low-Z detectors such as silicon and selenium.

Methods: A parallel-cascade approach is used to describe the three types of x-ray interactions. The description of incoherent scatter required developing expressions for signal and noise transfer through an “energy-labeled reabsorption” process where the parameters describing reabsorption are random functions of the scatter photon energy. The description of coherent scatter requires the use of scatter form factors that may not be accurate for some crystalline detector materials. The model includes the effects of scatter reabsorption and escape, charge collection, secondary quantum sinks, noise aliasing, and additive noise. Model results are validated by Monte Carlo calculations for Si and Se detectors assuming free-atom atomic form factors.

Results: The new signal and noise transfer expressions were validated by showing agreement with Monte Carlo results. Coherent and incoherent scatter can degrade the DQE of Si and sometimes Se detectors depending on detector thickness and incident-photon energy. Incoherent scatter can produce a substantial low-frequency drop in the modulation transfer function and DQE.

Conclusions: A generally useful CSA model of the DQE is described that is believed valid for any single-Z material up to 10 cycles/mm at both mammographic and radiographic energies within the limitations of Fourier-based linear-systems models and the use of coherent-scatter form factors. The model describes a substantial low-frequency drop in the DQE of Si systems due to incoherent scatter above 20–40 keV. © 2013 American Association of Physicists in Medicine. [http://dx.doi.org/10.1118/1.4794495]

Key words: cascaded-systems analysis (CSA), coherent scattering, incoherent scattering, photoelectric interaction, modulation transfer function (MTF), noise power spectrum (NPS), detective quantum efficiency (DQE), Monte Carlo simulation

I. INTRODUCTION

The signal and noise performance of x-ray imaging systems is generally described in terms of the detective quantum efficiency (DQE).1, 2 While the DQE is restricted to linear and shift invariant systems with only wide-sense stationary (WSS) or wide-sense cyclostationary (WSCS) noise processes3 and does not describe all aspects of system performance, it is safe to say that, with the exception of signal aliasing and some spectral considerations,4 the DQE is our primary metric of detector performance and the image quality is maximized for a given exposure only when the DQE is maximized.5 Expressing the DQE as a function of spatial frequency in the image plane accounts for spatial correlations in image signal and noise.

The DQE is a measure of the equivalent quantum efficiency of the system, defined as the ratio of the image noise-equivalent number of quanta (NEQ) to the true incident...
Due to the importance of the DQE for image quality and regulatory requirements, development of theoretical models is a critical step in detector design and assessment. Such models provide an understanding of the key physical parameters that determine image quality, and benchmarks against which performance can be evaluated. For example, linear-systems theory has been successful in describing systems having a deterministic response.\textsuperscript{6} Rabbani, Shaw and Van Metter\textsuperscript{7, 8} showed that MTF and signal-to-noise ratio (SNR) metrics could be expressed as transfer relationships through simple quantum-based gain and scatter processes, leading to an understanding of secondary quantum sinks\textsuperscript{5, 9–14} and the important impact they have on image quality. Yao and Cunningham\textsuperscript{15} extended this “cascaded-systems analysis” (CSA) to include parallel cascades and more comprehensive models that included fluorescence reabsorption and other considerations.\textsuperscript{16–20} Akbarpour, Friedman and co-workers extended the approach to include spatiotemporal effects in fluoroscopic detectors.\textsuperscript{21, 22}

These models describe signal and noise transfer through photoelectric interactions in the detector, neglecting the effect of incoherent (Compton) and coherent (Rayleigh) scatter. This is a reasonable simplification for many detectors of importance where the majority of all x-ray interactions are photoelectric, but is not adequate for low-Z detectors such as silicon (Si) and sometimes selenium (Se) as illustrated in Fig. 1. While coherent scatter does not deposit energy directly, low-Z detectors are generally designed to be relatively thick so as to achieve high quantum efficiency, and low-angle scatter before energy is deposited is a significant concern.\textsuperscript{23} Coherent and incoherent scatter both degrade the MTF\textsuperscript{24, 25} and the DQE, although the effect on DQE is more complex.

This paper describes an extension to the CSA approach that incorporates an approximate model of coherent and incoherent scatter. Although coherent scatter can give rise to interference effects and Bragg peaks (x-ray diffraction), particularly at lower x-ray energies,\textsuperscript{26, 27} this work assumes that scatter is adequately described by angle-dependent form factors. This assumption may fail in crystalline materials where interference can be very complicated and strongly dependent on angle of incidence. For higher energies and amorphous materials, the use of atomic form factors (AFFs) may be adequate. With this caveat, or under conditions where coherent scatter can be ignored, results are believed applicable for all

\begin{align}
\text{DQE}(k) &= \frac{\text{NEQ}(k)}{\overline{q}_{\text{in}}} = \frac{\overline{q}_{\text{in}} G^2 \text{T}^2(k)}{W(k)} = \frac{G^2 \text{T}^2(k)}{\overline{q}_{\text{in}} W(k)},
\end{align}

where $k$ denotes the spatial-frequency vector in cycles/mm, $\overline{q}_{\text{in}}$ is the number of photons per unit area incident on the detector, $G = \overline{d}/\overline{q}_{\text{in}}$ is the large-area detector gain, $\overline{d}$ is the average detector output signal, $T(k)$ is the system modulation transfer function (MTF), and $W(k)$ is the image Wiener noise power spectrum (NPS).
single-element detector materials at both mammographic and radiographic energies and validated for Si (Z = 14) and Se (Z = 34) detectors using Monte Carlo (MC) calculations assuming free-atom model (FAM) form factors for coherent scatter.

II. THEORY

Figure 1 is an illustration of the relative probability of photoelectric (PE), incoherent (INC), and coherent (COH) interactions as a function of energy and detector material.28 All three will degrade the MTF and DQE. In PE and INC interactions, energy is deposited at the interaction site and potentially at a remote site through reabsorption of a scatter photon. In COH, all energy is transferred to the scatter photon. One can therefore represent all three interaction types as special cases of a simple “generalized” interaction that consists of the creation and possible reabsorption of a scatter photon. In this work it is assumed that the scatter photon either escapes the detector or deposits all its energy at the reabsorption site. Ignoring multiple scatter where a second scatter photon may be emitted at the reabsorption site is justified on the grounds that it does not happen frequently, and when it does the second scatter has lower energy than the first and therefore normally travels a shorter distance. Consistent with previous work from our group,15,19 we make the following additional assumptions: (i) all detectors are linear and shift invariant and all noise processes are WSS; (ii) for PE interactions, we assume a single edge energy that may be an L or K edge depending on photon energy consistent with the simple atom model described by Hajdok et al.;19 (iii) results for a broad x-ray spectrum are obtained from a weighted average of monenergetic results;14 (iv) in the MC calculations, the number of secondaries (e-h pairs) produced is assumed proportional to the energy deposited in the x-ray convertor (deterministic gain) which ignores a potential secondary quantum sink in the number of liberated charges and is valid as long as many charges (e.g., >100)3 are collected per interacting x-ray photon; (v) all energy transferred to electrons is absorbed locally, i.e. electron-hole pair (ehp) transport considerations are ignored; (vi) the collection efficiency \( \beta \) of secondary quanta is independent of interaction depth; and (vii) incident photons are normal to the detector plane. Ignoring ehp transport considerations limits the spatial frequency range to approximately 10 cycles/mm.29

A schematic illustration of the CSA model is shown in Fig. 2. The input is a distribution of identical (same energy) incident x-ray quanta \( \tilde{q}_{\text{in}} \) [mm\(^{-2}\)]:

\[
\tilde{q}_{\text{in}}(r) = \sum_{n=1}^{\tilde{N}} \delta(r - \tilde{r}_n),
\]

where each \( \delta \)-function represents one incident photon, and \( \tilde{N} \) and \( \tilde{r}_n \) are random variables (RVs) describing the total number and image-plane position of each photon. A sparse sample input is illustrated by the rectangular image at the model input in Fig. 2 with dots representing incident photons. The purpose of the CSA model is to represent the subsequent processes converting this input to detector output signal values \( \tilde{d} \).

Each incident photon in \( \tilde{q}_{\text{in}} \) interacts with probability

\[
P(t) = \frac{\mu_{t}}{\mu} \left[ 1 - e^{-\mu L} \right]
\]

where \( t \) denotes interaction type (PE, COH, or INC), \( \mu = \mu_{\text{PE}} + \mu_{\text{INC}} + \mu_{\text{COH}} \), and \( L \) is the x-ray convertor thickness. The resulting distribution of secondary quanta (liberated charges) \( \tilde{q}_{\text{sec}} \) is obtained by summing contributions from each path.

II.A. Generalized x-ray interaction

Each shaded box in Fig. 2 is a special case of a generalized interaction model shown in Fig. 3. As illustrated in Fig. 3(a), an incident photon with energy \( E \) interacts in the x-ray convertor material and generates a scatter photon at scatter angle \( \theta \) and azimuthal angle \( \phi \) with respect to the incident path. Secondary quanta (light photons or e-h pairs) at selected stages of the model. Larger dots represent multiple secondary quanta at the same location following a quantum gain stage.

FIG. 2. Schematic illustration of the CSA model describing transfer of signal and noise through photoelectric, coherent, and incoherent interactions using a parallel-cascade model. Distributions \( \tilde{q}_{\text{in}}, \tilde{q}_{\text{sec}}, \) and \( \tilde{q}_{\text{col}} \) describe spatial distributions of incident x-ray quanta, liberated secondary quanta, and collected secondary quanta, respectively. The rectangular illustration at the model input represents a sparse sample distribution of identical incident x-ray quanta where each dot represents one photon. Dots in subsequent illustrations represent liberated secondary quanta (light photons or e-h pairs) at selected stages of the model.
electron-hole pairs in a photoconductor) are liberated at the primary interaction site and also the reabsorption site unless the scatter photon escapes the detector. A cascaded model of this process is shown in Fig. 3(b). While the model is similar to the simple-atom model of photoelectric interactions,\textsuperscript{19} it has been modified to accommodate incoherent scatter where parameters describing scatter production and reabsorption are random functions of the scatter-photon energy, which is itself a random function of incident photon energy. This is achieved by describing the shaded box in Fig. 3(b) as a single complex “energy-labeled scatter reabsorption” process. The resulting new signal and noise transfer relationships are developed in the Appendix using stochastic point-process theory.\textsuperscript{15,21,30}

The mean number of generated secondaries is given by

$$\overline{q}_s^{(t)} = \overline{q}_A + \overline{q}_B + \overline{q}_C$$ \hspace{1cm} (4)

for each interaction type, where the $t$ designation is dropped on the right-hand side for clarity, $\overline{q}_A = \overline{q}_{in} P_{01} (1 - S) \overline{g}_A$, $S$ is the probability of generating a scatter or emission photon, and $\overline{g}$ is the mean conversion gain determined by the deposited energy. Expressions for $\overline{q}_B$ and $\overline{q}_C$ are complicated since the scattered energy is a random variable, but are developed in the Appendix giving Eqs. (A11) and (A12) respectively. Combining these gives

$$\overline{q}_s^{(t)} = \overline{q}_{in} P_{01} [(1 - S) \overline{g}_A + S(\overline{g}_B)_s + (f \overline{g}_C)_s]$$ \hspace{1cm} (5)

for each interaction type, where $f$ represents the reabsorption probability and $\langle \rangle_s$ denote an average over the spectrum of scatter photons for INC and COH.\textsuperscript{14}

The MTF is given by a weighted average of contributions from each interaction type:

$$T^{(t)}(k) = \frac{(1 - S) \overline{g}_A + S(\overline{g}_B)_s + (f \overline{g}_C)_s}{(1 - S) \overline{g}_A + S(\overline{g}_B)_s + (f \overline{g}_C)_s}$$ \hspace{1cm} (6)

where $R$ is the normalized reabsorption characteristic function (RCF) describing scatter/emission photon relocation as given by the Fourier transform of the reabsorption point spread function (PSF).\textsuperscript{19}

Path A is not correlated with B or C and so there is only the one cross term, between B and C as given by Eq. (A32). The NPS associated with $\overline{q}_s^{(t)}$ is therefore\textsuperscript{15,19}

$$W_s^{(t)}(k) = W_A(k) + W_B(k) + W_C(k) + 2 W_{BC}(k)$$ \hspace{1cm} (7)

for each interaction type where $W_A(k) = \overline{q}_{in} P_{01} (1 - S) (\overline{g}_A^2 + \sigma_{\overline{g}_A}^2)$, $\sigma_{\overline{g}_A}^2$ is the variance in $\overline{g}_A$ and the sum $W_B + W_C + 2W_{BC}$ is given by Eq. (A33). Therefore,

$$W_s^{(t)}(k) = \overline{q}_{in} P_{01} (1 - S) \left[ \overline{g}_B^2 + \sigma_{\overline{g}_B}^2 \right] + S \left[ \overline{g}_A^2 + \sigma_{\overline{g}_A}^2 \right] + (f \overline{g}_C + \sigma_{\overline{g}_C}^2) + 2 (f \overline{g}_B \overline{g}_C R(k))$$ \hspace{1cm} (8)

Table I gives gain parameters used for each interaction type where $P_K$, $\omega_K$, $E_K$, $E'$, $w$, and $f$ are K-shell participation fraction, fluorescence yield, fluorescent photon energy, INC scatter-photon energy, effective energy required to liberate one secondary pair, and scatter/emission reabsorption probability, respectively.

### II.B. Photoelectric, incoherent, and coherent interactions

#### II.B.1. Photoelectric interaction

In a PE event, path A corresponds to those events that do not produce a fluorescent photon and incident energy $E$ is assumed to be completely deposited at the primary interaction site, liberating $\overline{g}_A$ secondaries (see Table I). Paths B and C describe events with emission of a fluorescent photon, resulting in $\overline{g}_B$ secondaries liberated locally per PE event and $\overline{g}_C$ liberated remotely with probability $f_k$. Since the fluorescent photon energy is fixed, we can use Eqs. (5) to (8) and write the mean, MTF and NPS corresponding to photoelectric interactions as

$$\overline{q}_s^{(pe)} = \overline{q}_{in} P_{pe} [(1 - S) \overline{g}_A + S(\overline{g}_B + f_k \overline{g}_C)]$$ \hspace{1cm} (9)

$$T^{(pe)}(k) = \frac{(1 - S) \overline{g}_A + S(\overline{g}_B + f_k \overline{g}_C R(k))}{(1 - S) \overline{g}_A + S(\overline{g}_B + f_k \overline{g}_C)}$$ \hspace{1cm} (10)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PE</th>
<th>COH</th>
<th>INC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$P_{K,0K}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\overline{q}_A$</td>
<td>$E/w$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\overline{q}_B$</td>
<td>$(E - E_K)/w$</td>
<td>0</td>
<td>$(E - E')/w$</td>
</tr>
<tr>
<td>$\overline{q}_C$</td>
<td>$E_K/w$</td>
<td>$E/w$</td>
<td>$E'/w$</td>
</tr>
</tbody>
</table>

---

**Fig. 3.** Schematic illustration of the “generic interaction” used in this work showing: (a) incident x-ray photon interacting in the x-ray converter and producing a scatter photon at polar angle $\theta$ and azimuthal angle $\phi$; and (b) CSA model. The three paths in (b) represent events that liberate charges from: (path A) primary interaction site when no scatter/emission photon is generated; (path B) primary interaction site when a scatter/emission photon is generated; and (path C) remote reabsorption of scatter/emission photon.
II.B.2. Coherent scatter

In COH events, low-angle scatter can result in interference effects and, in the case of crystalline materials, Bragg peaks. At high values of the momentum-transfer argument $\chi$, where $\chi = \frac{E}{h c} \sin(\theta/2)$ and $hc$ is the product of Planck’s constant and the speed of light, the FAM of scatter is a good approximation and the scatter cross section is the product of the Thomson cross section (unpolarized) describing scatter from free electrons and the squared AFF describing interference of scatter from atomic electrons. Compound materials are described using a combination of AFFs using the independent atom model (IAM) approximation. However, the FAM assumption fails in two important situations: at energies near atomic edge energies (anomalous scatter) and in crystalline materials at low momentum transfer ranges due to both constructive and destructive interference effects from periodic crystal structure.

At frequencies up to 10 cycles/mm, this corresponds to $\chi \gtrsim 0.01$ in Si and $\chi \gtrsim 0.2$ $\text{Å}^{-1}$ in Se with the difference being due to the relatively low density and therefore greater converter thickness required for Si. The impact of coherent scatter is generally unimportant in higher-Z materials due to greater mass density. The FAM approach is generally accurate for $\chi > 0.1$ $\text{Å}^{-1}$ for amorphous materials such as Pb, Hg, and even powdered Si, and for $\chi > 2$ $\text{Å}^{-1}$ for crystalline materials. In the range $0.01 < \chi < 2$ $\text{Å}^{-1}$, there is wide-spread use of molecular or other empirical form factors to describe scatter amplitude although use of form factors is an approximation that fails at very high energies (>1 MeV) and in crystalline materials with strong diffraction effects at very low momentum-transfer values.

The CSA model provides a good description of the MTF and DQE only when form factors are appropriate. When form factors are inaccurate or not known, this approach must be viewed as an approximation. However, this approximation is likely reasonable since coherent scatter accounts for a relatively small fraction of all x-ray interactions and hence errors in this assumption have a proportionally smaller effect on the overall DQE. In addition, since the MTF is degraded by the average of many coherent-scatter interactions, it is primarily the integral scatter cross section and range of scatter angles that are important for this application, and the form-factor approach usually gives a good estimate of these.

The COH event itself does not deposit energy. Rather, a scatter photon with the same energy can be absorbed with probability $f_{\text{coh}}$ that is described by path C or escape with probability $(1 - f_{\text{coh}})$. If absorbed, we assume all energy $E$ is deposited, corresponding to a mean gain $g_{\text{C}} = E/w$ with characteristic function $R_{\text{coh}}$. Using Eqs. (5) to (8), the mean number of quanta, MTF and NPS due to reabsorption of coherent-scatter photons is given by

$$W_{\text{sec}}^<(k) = \bar{q}_{\text{in}}P_{\text{coh}}[1 - S(\vec{g}_B^2 + \sigma_{gB}^2) + S\{\vec{g}_B^2 + \sigma_{gB}^2\}
+ f_{\text{coh}}(\vec{g}_C^2 + \sigma_{gC}^2) + 2f_{\text{coh}}\vec{g}_BR_{\text{coh}}(k)]], \quad (11)$$

$$T_{\text{coh}}(k) = R_{\text{coh}}(k), \quad (13)$$

$$W_{\text{sec}}^>(k) = \bar{q}_{\text{in}}P_{\text{coh}}f_{\text{coh}}(\vec{g}_C^2 + \sigma_{gC}^2). \quad (14)$$

II.B.3. Incoherent scatter

In an INC event, the incident photon interacts with a loosely bound (free) electron producing a Compton photon and recoil electron. The energy of the Compton photon $E'$ is determined by incident energy $E$ and scatter polar angle $\theta$:

$$E' = \frac{E}{1 + \alpha(1 - \cos \theta)}, \quad (15)$$

where $\alpha = E/m_e c^2$ represents the incident photon energy in units of the electron rest-mass energy ($m_e c^2 \simeq 511$ keV). The recoil electron transfers its energy at the primary interaction site with a mean conversion gain of $(E - E')/w$, and the Compton photon is reabsorbed after relocation with a probability $f_{\text{inc}}$, characteristic function $R_{\text{inc}}$, and conversion gain $E'/w$. The mean number of quanta, MTF and NPS of INC interactions are therefore given by

$$\bar{q}_{\text{sec}}^{(\text{inc})} = \bar{q}_{\text{in}}P_{\text{inc}}\langle \vec{g}_B \rangle + \langle f_{\text{inc}}\vec{g}_C \rangle, \quad (16)$$

$$T_{\text{inc}}(k) = \frac{\langle \vec{g}_B \rangle + \langle f_{\text{inc}}\vec{g}_C R_{\text{inc}}(k) \rangle}{\langle \vec{g}_B \rangle + \langle f_{\text{inc}}\vec{g}_C \rangle}, \quad (17)$$

$$W_{\text{sec}}^{(\text{inc})}(k) = \bar{q}_{\text{in}}P_{\text{inc}}\{\vec{g}_B^2 + \sigma_{gB}^2\rangle + \langle f_{\text{inc}}\vec{g}_C^2 + \sigma_{gC}^2 \rangle
+ 2\langle f_{\text{inc}}\vec{g}_B R_{\text{inc}}(k) \rangle\}, \quad (18)$$

Since the INC scattered photon energy $E'$ has a spectrum determined by $E$ and $\theta$, averaging over the distribution of scatter energy is required. The probability density function (PDF) of INC scattered angle is introduced in Eq. (34). Equations (9) to (16) describe transfer of signal and noise through the three interaction types and together are used to construct DQE models of x-ray detectors as illustrated in Fig. 2.

II.C. Detector signal

The mean number of liberated secondary quanta that are collected and contribute to the detector signal is given by

$$\bar{q}_{\text{col}} = \beta \bar{q}_{\text{sec}} = \beta(\bar{q}_{\text{sec}}^\text{pe} + \bar{q}_{\text{sec}}^{\text{coh}} + \bar{q}_{\text{sec}}^{\text{inc}}), \quad (19)$$

where $\beta$ is the coupling efficiency. The final expression of detector signal after aperture integration, sampling, and additive noise process is given by

$$\bar{I} = \lambda a_x a_y \bar{q}_{\text{col}}, \quad (20)$$

where $\lambda$ is a constant of proportionality and $a_x \times a_y$ is the pixel aperture size representing the sensitive area in a detector element.
II.D. Modulation transfer function

The overall MTF is given by

$$
T(k) = \frac{1}{\bar{T}_\text{sec}} \left[ \bar{T}_\text{sec} T_{\text{pe}}(k) + \bar{T}_\text{sec} T_{\text{coh}}(k) + \bar{T}_\text{inc} T_{\text{inc}}(k) \right]
$$

$$
\times T_{\text{sec}}(k) T_{\alpha}(k),
$$

where \( T_{\text{sec}} \) describes the scatter of secondary quanta and \( T_{\alpha}(k) = \text{sinc}(\alpha k) \) as illustrated in Fig. 2.

II.E. Noise power spectrum

The NPS associated with \( \bar{T}_\text{col} \), the distribution of collected secondaries, is given by

$$
\bar{W}_{\text{col}}(k) = \beta^2 \left[ \bar{T}_{\text{sec}}(k) - \bar{T}_\text{sec}^2(k) + \beta \bar{T}_\text{sec} \right],
$$

where $W_{\text{sec}}(k) = W_{\text{sec}}^\text{pe}(k) + W_{\text{sec}}^\text{coh}(k) + W_{\text{inc}}^\text{inc}(k)$.

The image NPS in the \( x \)-direction, including aperture integration, sampling, and additive noise, is given by

$$
W(k_x) = \lambda^2 \bar{a}_x^2 \left[ \bar{W}_{\text{col}}(k_x) T_{\alpha}(k_x) + \sum_{n=1}^{\infty} \left\{ \bar{W}_{\text{col}}(k_x) \frac{\bar{T}_{\alpha}(k_x)}{k_x} \right\} \right],
$$

$$
+ 2 \bar{x}_o \sigma_{\text{add}},
$$

where \( k_x \) and \( x_o \) are the frequency vector and detector pixel pitch in the \( x \)-direction and \( \sigma_{\text{add}} \) represents the pixel standard deviation from additive electronic noise. The detector element fill factor is given by \( a_x/x_o \) in the \( x \)-direction.

II.F. Detective quantum efficiency

The detector DQE determined by the CSA model illustrated in Fig. 2 is then given by

$$
\text{DQE}(k) = \frac{(\bar{T}(k))^2_{\text{pe}}}{(\bar{T}_{\text{inc}})^2_{\text{pe}} (W(k))_{\text{pe}}},
$$

while our model assumed an incident photon with a single energy, it is applicable to the primary spectrum \( p \) by averaging over all spectral energies.

II.G. Reabsorption probabilities and characteristic functions

In this model, RCF \( R \) and reabsorption probability \( f \) are used to characterize the scatter/emission photon:

$$
f = CTF(k) |_{k=0} \quad \text{and} \quad R(k) = \frac{CTF(k)}{CTF(k) |_{k=0}},
$$

for each interaction type. The \( CTF(k) \) is the characteristic function describing reabsorption, averaged over four RVs \( \phi, l, \theta, \) and \( z \) as illustrated in Fig. 3(a) where \( \phi (0 \leq \phi \leq 2\pi) \) is the azimuthal angle, \( \theta (0 \leq \theta \leq \pi) \) is the polar angle, \( l \) is the reabsorption distance, and \( z (0 \leq z \leq L) \) is the interaction depth. The PDF associated with \( \phi \) is given by \( p(\phi) = 1/2\pi \) for all interactions corresponding to an isotropic emission in \( \phi \).

Detailed derivations are described by Hajdok et al.\(^{19}\) for PE fluorescence reabsorption, giving

$$
p_{\text{pe}}(l) = \mu_{\text{pe}}(E_K) e^{-\mu_{\text{pe}}(E_K) l},
$$

$$
p_{\text{pe}}(z) = \mu_{\text{pe}}(E) e^{-\mu_{\text{pe}}(E) z} / [1 - e^{-\mu_{\text{pe}}(E) L}],
$$

$$
p_{\text{pe}}(\theta) = (\sin \theta) / 2.
$$

Reabsorption of COH scatter photons corresponds to\(^{29}\)

$$
p_{\text{coh}}(l) = \mu_{\text{coh}}(E) e^{-\mu_{\text{coh}}(E) l},
$$

$$
p_{\text{coh}}(z) = \mu_{\text{coh}}(E) e^{-\mu_{\text{coh}}(E) z} / [1 - e^{-\mu_{\text{coh}}(E) L}],
$$

$$
p_{\text{coh}}(\theta) = \frac{1}{\sigma_{\text{coh}}^2} \left\{ \frac{r_0}{2} (1 + \cos^2 \theta) [F(\chi, Z)]^2 \right\} 2\pi \sin \theta,
$$

where \( p_{\text{coh}}(\theta) \) describes the low-angle scatter based on a form factor description \( F \) (see Fig. 4).\(^{32,37,38}\) It if is known that the

![Fig. 4. Angular PDFs for coherent scatter photons in Si (Z = 14) and Se (Z = 34) at (a) mammographic and (b) radiographic energy calculated using Eq. (32) which assumes FAM approximation.](Medical Physics, Vol. 40, No. 4, April 2013)
FAM is valid, the total coherent cross section per atom $\sigma_{coh}$ is given by the Thomson cross section (unpolarized) describing scatter from free electrons and the squared form factor:31, 39

$$\sigma_{coh} = \pi r_0^2 \int_0^{\pi} \sin \theta (1 + \cos^2 \theta)[F(\chi, Z)]^2 d\theta,$$

where $r_0 = 2.818 \times 10^{-13}$ cm is the classical electron radius. Compound materials are described using a combination of AFFs using the IAM approximation.26 In more complicated systems, molecular form factors may be required. The user must ensure appropriate use of form factors or view the model as an approximation that may not give an accurate description of coherent scatter.

The PDFs for the RVs associated with reabsorption of Compton scatter are

$$p_\text{inc}(l) = \mu_{\text{pe}}(E')e^{-\mu_{\text{pe}}(E')l},$$

$$p_\text{inc}(z) = \mu_{\text{inc}}(E)e^{-\mu_{\text{inc}}(E)z}/[1 - e^{-\mu_{\text{inc}}(E)L}],$$

where $\mu_{\text{pe}}(E')$ and $\mu_{\text{inc}}(E)$ are the photoelectric and total linear attenuation coefficients, respectively.

**Table II.** X-ray interaction related physical parameters for Si and Se used in this study (Attix Ref. 36).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Si</th>
<th>Se</th>
</tr>
</thead>
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<td>$\omega_K$</td>
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**FIG. 5.** Schematic illustration to describe the MC simulations of 2D point spread function and noise-field image which involve the “x-ray interaction physics” between incident x-ray and convertor materials. The particle tracking function of MCNP5TM (ORNL, USA) has been used to obtain the position and deposited energies due to each interaction event.

**FIG. 6.** Reabsorption probabilities of fluorescent and scatter photons as a function of incident photon energy with Si (left) and Se (right) for each interaction type.
\[ p_{\text{inc}}(\theta) = \frac{1}{e\sigma_{\text{inc}}} \left\{ \frac{r_0^2}{2} (1 + \cos^2 \theta) F_{\text{KN}}(\theta) \right\} 2\pi \sin \theta, \quad (36) \]

where \( p_{\text{inc}}(\theta) \) describes the distribution of scatter angles,\(^2^9\) the Compton cross section per electron is given by\(^4^0\)

\[ e\sigma_{\text{inc}} = 2\pi r_0^2 \left\{ \frac{1 + \alpha}{\alpha^2} \left[ \frac{2(1 + \alpha)}{1 + 2\alpha} - \frac{1}{\alpha} \ln(1 + 2\alpha) \right] + \frac{1}{2\alpha} \ln(1 + 2\alpha) - \frac{1 + 3\alpha}{(1 + 2\alpha)^2} \right\}, \quad (37) \]

and the Klein-Nishina factor \( F_{\text{KN}}(\theta) \), is given by\(^2^9\)

\[ F_{\text{KN}}(\theta) = \frac{1}{[1 + \alpha(1 - \cos \theta)]^2} \times \left\{ 1 + \frac{\alpha^2(1 - \cos \theta)^2}{(1 + \cos^2 \theta)[1 + \alpha(1 - \cos \theta)]} \right\}. \quad (38) \]

Combining the above expressions gives a comprehensive analytic model of the DQE of a large-area single-Z detector at mammographic and diagnostic energies and includes the effects of x-ray scatter, secondary quantum sinks, noise aliasing, and additive noise. The model does not include the effect of electron transport which may be important at frequencies greater than 10 cycles/mm.\(^1^9\)

### III. VALIDATION

The model was validated with a Monte Carlo study for Si and Se x-ray convertor materials which generally have the highest levels of incoherent scatter. Thicknesses appropriate for mammography and radiography were chosen as 4 and 7 mm for Si and 0.2 and 0.5 mm for Se. Table II summarizes some physical parameters used in the model.

#### III.A. Monte Carlo simulation

Monte Carlo simulations were used to validate the new descriptions of signal and noise through the “energy-labeled re-absorption” process. The particle tracking (pTrac) tally of MCNP5\textsuperscript{TM} (ORNL, USA) was used as illustrated in Fig. 5. Electron transport considerations were disabled as these are known to be unimportant at spatial frequencies below 10 cycles/mm at radiographic energies.\(^1^9\) AFFs and the free-atom model were assumed.

The 2D PSF was determined using a zero-width pencil beam with \(10^7\) histories and 2 \(\mu\)m square scoring bins covering an area of 4.0\(\times\)4.0 mm. After verifying the PSF to be
IV. RESULTS

IV.A. Reabsorption probability

Figure 6 shows reabsorption probabilities for the scatter/emission photon as a function of incident photon energy with Si and Se for each interaction type. Unlike $f_{inc}$ and $f_{coh}$, $f_{K}$ remains large at high photon energies due to the fixed fluorescence energy. The value of $f_{coh}$ is lower than that of $f_{inc}$ due to COH scatter photons having a higher energy.

IV.B. Reabsorption characteristic function

The RCFs of x-ray interactions for Si and Se x-ray converters are shown in Fig. 7 for different thicknesses and incident photon energies. While the probability of reabsorption is greater for INC than COH due to lower photon energy, the characteristic function decreases more quickly with frequency due to the lower energy and greater scatter angle.

IV.C. Liberated secondaries

The number of liberated secondaries per incident photon predicted by the CSA model, $q_{sec}/q_{in}$, shows excellent agreement with the MC result as illustrated in Fig. 8. The analytic model shows that INC interactions are responsible for most of the liberated charges in Si while PE interactions dominate in Se.

IV.D. Modulation transfer function

Figures 9 and 10 show a comparison of the analytic and MC MTFs under various conditions. In general, INC
interactions result in a substantial low-frequency drop of the MTF while PE interactions cause a slower decrease with frequency. The total MTF shows excellent agreement with MC simulations in every case.

In Si, PE effects on the MTF are small due to the low fluorescent energy at $\sim$1.7 keV while COH has an important influence at 20 keV (mammography) but not at higher energies.

### IV.E. Noise power spectrum

Figure 11 shows a comparison of the NPS normalized by $P_2$ obtained with the CSA model and MC simulations. Excellent agreement is shown in all cases although there may be a very small difference at 80 keV in Si, thought to be due to the CSA model ignoring multiple scatter events. Uncertainty in the NPS decreases with increasing frequency due to
IV.F. Detective quantum efficiency

Figure 12 shows a comparison of the analytic DQE with MC results. In all cases tested, excellent agreement was achieved with both assuming FAM form factors. The importance of including INC and COH interactions in CSA models for Si detectors is illustrated in Fig. 13(a). Also shown is the degradation in DQE that can be expected with poor charge collection, resulting in a secondary quantum sink as illustrated in Fig. 13(b), and suggesting minimum acceptable collection efficiencies may be 1% and 10% for Si and Se respectively at 20 keV. While results are dependent on detector particulars and possibly additive noise, they highlight the critical role of these parameters and the importance of theoretical models.

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Fig. 11. Comparison of NPS curves normalized by $d^2$, obtained from the CSA model with MC calculations. While excellent agreement was obtained in all cases, it is thought the CSA results are slightly less than MC results at 80 keV in Si due to the CSA model ignoring multiple scattering events.

Fig. 12. Comparison of DQE curves obtained from the CSA model with MC calculations. Excellent agreement is seen in all cases considered.
V. CONCLUSIONS

An important contribution of this work is the development of transfer relationships through what we have called an energy-labeled reabsorption process with similarities to Rabbanii and Van Metter. This leads to transfer expressions for a cascaded-systems model of the DQE that is applicable to a wide spectrum of single-Z detector materials. The model is believed accurate up to approximately 10 cycles/mm (limited by absence of electron-hole pair transport considerations) and assumes a form-factor based description of coherent scatter that may not be accurate in crystalline detector materials. It is validated by Monte Carlo simulations for Si and Se detectors.

It is shown both coherent and incoherent scatter can have substantial effects on the MTF and DQE of Si detectors, depending on Si thickness, often seen as a low-frequency drop in each. The CSA model quantifies this degradation and provides insight into novel detector designs such as slot-scanning mammography and photon-counting detectors. Neither coherent nor incoherent scatter are important in Se detectors at mammographic energies, although incoherent scatter degrades the MTF by approximately 5% at 80 keV.

An important use of CSA models is to determine the impact of various detector parameters on the DQE. This work shows that the collection efficiency (of liberated charge pairs) must be greater than 1% in Si and 10% in Se to avoid degradation of the DQE at 20 keV even without additive noise. The CSA model includes the effects of photoelectric, coherent and incoherent interactions, secondary quantum sinks, charge collection efficiency, noise aliasing and additive noise, but does not include nonlinear or depth-dependent interaction and charge-collection processes (Lubberts effects), electron transport (important above 10 cycles/mm) or detector materials with more than one atomic element.

ACKNOWLEDGMENTS

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APPENDIX: ENERGY-LABELED SCATTER AND REABSORPTION PROCESS

Figure 14 shows a schematic block diagram of the energy-labeled scatter and reabsorption process where each parameter is a random function of the scatter-photon energy $E'$. The input to this process $\gamma$ describes locations of events that produce scatter photons, represented by the random distribution of Dirac $\delta$ functions:

$$\gamma_r = \sum_{n=1}^{N} \delta (r - r_n),$$

(A1)

where $\tilde{N}$ and $\tilde{r}_n$ describe the total number and image-plane positions of one interaction type (PE, INC, or COH). Path B describes the liberation of secondary quanta ($e^-h$ pairs) at the interaction site and path C describes the liberation of secondary quanta resulting from reabsorption of the scatter photon at a near-by location. The output distribution $\tilde{\gamma}_{B+C}$ gives the total distribution of liberated quanta:

$$\tilde{\gamma}_{B+C}(r) = \sum_{n=1}^{\tilde{N}} \left[ \tilde{g}_{B}(\tilde{E}_n') \delta (r - \tilde{r}_n) + \tilde{f}_n(\tilde{E}_n') \tilde{g}_{C}(\tilde{E}_n') \delta (r - \tilde{r}_n - \Delta \tilde{r}_n(\tilde{E}_n')) \right],$$

(A2)

![Fig. 14. Schematic illustration of the coupled reabsorption process used for the energy-labeled x-ray interaction process.](image)
where $\tilde{g}_n$ is the conversion gain of the $n$th interaction in each path, $f_n$ is a Bernoulli RV (sample values of 0 or 1 only) describing whether the $n$th scatter photon is reabsorbed, and $\Delta\tilde{r}_n$ is a random vector describing where the scatter photon deposits its energy. The corresponding autocovariance associated with the output is given by

$$K_{BC}(\mathbf{r}, \mathbf{r}') = K_B(\mathbf{r}, \mathbf{r}') + K_C(\mathbf{r}, \mathbf{r}') + 2K_{BC}(\mathbf{r}, \mathbf{r}'), \quad (A3)$$

where $K_{BC}$ is the cross-covariance term.

Variables $f_n$, $\Delta\tilde{r}_n$, $g_B$, and $g_C$ are energy labeled as they are random functions of the scatter-photon energy $E'$, similar to the input-labeled process described by Rabbani and Van Metter.\textsuperscript{8} The mean and autocovariance are determined here by averaging over each RV similar to the approach of Barrett et al.\textsuperscript{30}

1. Mean and covariance of input

The mean and autocovariance of the input $\tilde{g}_n$ will be required below. General expressions for the mean and autocorrelation of a random point process have been described previously by Akbarpour et al.\textsuperscript{21} as

$$\tilde{g}_n(\mathbf{r}) = \overline{N} p(\mathbf{r})$$

$$R_n(\mathbf{r}, \mathbf{r}') = \overline{N}\delta(\mathbf{r} - \mathbf{r}') p(\mathbf{r}) + (\overline{N}^2 - \overline{N} + \sigma_N^2) p(\mathbf{r}) p(\mathbf{r}'), \quad (A4)$$

where $\sigma_N^2$ is the variance in $\overline{N}$ and $p(\mathbf{r})$ is the position probability density of scatter events. The autocovariance of $g_n$ is therefore given by

$$K_n(\mathbf{r}, \mathbf{r}') = R_n(\mathbf{r}, \mathbf{r}') - \tilde{g}_n(\mathbf{r})\tilde{g}_n(\mathbf{r}'). \quad (A5)$$

2. Mean of output

We start with the second term in Eq. (A2) corresponding to path C. Averaging over $\overline{g}_C n$ for given $\overline{E}_n$ gives

$$E[\overline{g}_C(\mathbf{r})] = \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n')$$

$$= \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n') \langle \delta(\mathbf{r} - \overline{r}_n - \Delta\overline{r}_n(\overline{E}_n')) \rangle, \quad (A6)$$

where $\overline{g}_n(\overline{E}_n') = E[\overline{g}_n(\overline{E}_n')|\overline{E}_n']$. Averaging over $\Delta\overline{r}_n$ for given $[\overline{r}_n, \overline{E}_n']$ gives

$$E[\overline{g}_C(\mathbf{r})] = \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n')$$

$$\times \int_A \delta(\mathbf{r} - \overline{r}_n - \Delta\overline{r}_n(\overline{E}_n'))$$

$$\times p(\Delta\overline{r}_n(\overline{E}_n')|\overline{E}_n') d\overline{r}_n$$

$$= \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n') p(\Delta\overline{r}_n|\overline{E}_n'), \quad (A7)$$

where $p(\Delta\overline{r}_n)$ is the scatter PDF. Averaging over $\overline{r}_n$ for given $\overline{E}_n'$ gives

$$E[\overline{g}_C(\mathbf{r})] = \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n')$$

$$\times \int_A p(\Delta\overline{r}_n(\mathbf{r} - \overline{r}_n|\overline{E}_n') p(\overline{r}_n|\overline{E}_n') d\overline{r}_n$$

$$= \sum_{n=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n') \int_A, \quad (A8)$$

where $A$ is the detector area and $\int_A p(\Delta\overline{r}_n(\mathbf{r} - \overline{r}_n|\overline{E}_n') p(\overline{r}_n|\overline{E}_n') d\overline{r}_n$ reduces to $1/A$ for every energy $\overline{E}_n'$. Averaging over $\overline{E}_n'$ gives

$$E[\overline{g}_C(\mathbf{r})] = \frac{1}{A} \sum_{n=1}^{\overline{N}} \int_E f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n') p(\overline{E}_n') d\overline{E}_n'$$

$$= \frac{1}{A} \sum_{n=1}^{\overline{N}} (f_n\overline{g}_C)n s, \quad (A9)$$

where $s(E')$ is the normalized spectrum of scatter-photon energies and angle brackets $\langle \rangle$ denote an average over $s(E')$. Noting that $\langle f_n\overline{g}_Cn \rangle$ is the same for every incident photon having energy $E$ and averaging over $\overline{N}$ gives

$$\overline{g}_C(\mathbf{r}) = \frac{\overline{N}}{A} \langle f\overline{g}_C \rangle_s = \overline{g}_s(\langle f\overline{g}_C \rangle_s). \quad (A10)$$

A similar derivation for path B gives

$$\overline{g}_B(\mathbf{r}) = \overline{g}_s(\langle f\overline{g}_B \rangle_s). \quad (A12)$$

The sum of Eqs. (A10) and (A12) gives the mean distribution of scatter photons in Eq. (4).

3. Autocovariance and Wiener NPS of output

The autocovariance is determined following a similar approach although fewer steps are shown. The autocorrelation of path C is given by

$$R_C(\mathbf{r}, \mathbf{r}') = E[\overline{g}_C(\mathbf{r})\overline{g}_C(\mathbf{r}')]$$

$$= \sum_{n=1}^{\overline{N}} \sum_{m=1}^{\overline{N}} f_n(\overline{E}_n')\overline{g}_Cn(\overline{E}_n') f_m(\overline{E}_m')\overline{g}_Cm(\overline{E}_m')$$

$$\times \delta(\mathbf{r} - \overline{r}_n - \Delta\overline{r}_n(\overline{E}_n'))$$

$$\times \delta(\mathbf{r}' - \overline{r}_m - \Delta\overline{r}_m(\overline{E}_m')) \bigg\rangle, \quad (A13)$$

It is convenient to separate the above terms for which $n = m$ and $n \neq m$. For $n = m$, averaging over $\overline{g}_Cn$ gives
\[ R_C(r, r')|_{\vec{r}_n} = \sum_{n=1}^{N} \tilde{f}_n (E_n) g_{C_n}(\tilde{E}_n) \times \delta(r - \tilde{r}_n - \Delta \tilde{r}_n(\tilde{E}_n)) \times \delta(r' - \tilde{r}_n - \Delta \tilde{r}_n(\tilde{E}_n)). \] (A14)

Averaging over \( \Delta \tilde{r}_n \) gives
\[ R_C(r, r')|_{\vec{r}_n} = \sum_{n=1}^{N} \tilde{f}_n (E_n) g_{C_n}(\tilde{E}_n) \times \delta(r - r') \rho_{\Delta r}(r - \tilde{r}_n)|E_n\). \] (A15)

Averaging over \( \vec{r}_n \) gives
\[ R_C(r, r')|_{\vec{r}_n} = \sum_{n=1}^{N} \tilde{f}_n (E_n) g_{C_n}(\tilde{E}_n) \delta(r - r') / A. \] (A16)

Averaging over \( \tilde{r}_n \) gives
\[ R_C(r, r')|_{\vec{r}_n} = \sum_{n=1}^{N} \tilde{f}_n (E_n) g_{C_n}(\tilde{E}_n) \delta(r - r') / A, \] (A17)
where \( \tilde{f} = \tilde{f} \) for the Bernoulli RV. Averaging over \( E' \) and then \( \tilde{N} \) and then substituting \( \sigma^2 = g^2 - \tilde{g}^2 \) gives
\[ R_C(r, r')|_{\vec{r}_n} = \tilde{g}_d \delta(r - r')(f \tilde{g}_C^2 + f \sigma^2_{\tilde{g}_C})/s. \] (A18)

For \( n \neq m \), averaging over \( g_{C_n} \) gives
\[ R_C(r, r')|_{n \neq m} = \sum_{n=1}^{N} \sum_{m=1}^{N} \tilde{f}_n (E_n) \tilde{f}_m (E_m) \times g_{C_n}(\tilde{E}_n) g_{C_m}(\tilde{E}_m) \times \delta(r - \tilde{r}_n - \Delta \tilde{r}_n(\tilde{E}_n)) \times \delta(r' - \tilde{r}_m - \Delta \tilde{r}_m(\tilde{E}_m)). \] (A19)

Averaging over \( \Delta \tilde{r}_n \) gives
\[ R_C(r, r')|_{n \neq m} = \sum_{n=1}^{N} \sum_{m=1}^{N} \tilde{f}_n (E_n) \tilde{f}_m (E_m) \times g_{C_n}(\tilde{E}_n) g_{C_m}(\tilde{E}_m) \times \rho_{\Delta r}(r - \tilde{r}_n)|E_n| \times \rho_{\Delta r}(r' - \tilde{r}_m)|E_m|). \] (A20)

Averaging over \( \tilde{r}_n \) gives
\[ R_C(r, r')|_{n \neq m} = \sum_{n=1}^{N} \sum_{m=1}^{N} \tilde{f}_n (E_n) \tilde{f}_m (E_m) \times g_{C_n}(\tilde{E}_n) g_{C_m}(\tilde{E}_m) \times \int_A \int_A p_{\Delta r}(r - r_n)|E_n| p_{\Delta r}(r' - r_m)|E_m|) \times p(r_n|E_n)p(r_m|E_m)d^2r_nd^2r_m. \] (A21)

Averaging over \( \tilde{f}_n \) gives
\[ R_C(r, r')|_{n \neq m} = \sum_{n=1}^{N} \sum_{m=1}^{N} \tilde{f}_n (E_n) \tilde{f}_m (E_m) \times g_{C_n}(\tilde{E}_n) g_{C_m}(\tilde{E}_m) \times \int_A \int_A p_{\Delta r}(r - r_n)|E_n| p_{\Delta r}(r' - r_m)|E_m|) \times p(r_n|E_n)p(r_m|E_m)d^2r_nd^2r_m. \] (A22)

Averaging over \( E' \) and then \( \tilde{N} \) gives
\[ R_C(r, r') = (\tilde{N}^2 - \tilde{N}) \int_{E'} \int_{E'} f(E_n)f(E_m)g_{C_n}(E_n)g_{C_m}(E_m) \times \int_A \int_A p_{\Delta r}(r - r_n|E_n)p_{\Delta r}(r' - r_m|E_m) \times p(r_n|E_n)p(r_m|E_m)d^2r_nd^2r_m. \] (A23)

Finally, adding the results for the two cases and combining with Eq. (A4) to express \( R_C(r, r') \) in terms of the autocorrelation of the input where \( \tilde{N}^2 = \sigma^2_N + \tilde{N}^2 \) gives:
\[ R_C(r, r') = \tilde{g}_d \delta(r - r')(f \tilde{g}_C^2 + f \sigma^2_{\tilde{g}_C})/s + \int_{E'} \int_{E'} f(E_n)f(E_m)g_{C_n}(E_n)g_{C_m}(E_m) \times \int_A \int_A p_{\Delta r}(r - r_n|E_n)p_{\Delta r}(r' - r_m|E_m) \times |R_s(r_n, r_m) - \tilde{g}_d \delta(r_n - r_m)|d^2r_nd^2r_m \times p(E_n)p(E_m)dE_ndE_m. \] (A24)

The autocovariance of path \( C \) is therefore \( K_C(r, r') = R_C(r, r') - \tilde{g}_d \delta(r - r')g_{C}(r') \) and combining with Eqs. (A5) and (A24) gives
\[ K_C(r, r') = \tilde{g}_d \delta(r - r')(f \tilde{g}_C^2 + f \sigma^2_{\tilde{g}_C}) + \int_{E'} \int_{E'} f(E_n)f(E_m)g_{C_n}(E_n)g_{C_m}(E_m) \times \int_A \int_A p_{\Delta r}(r - r_n|E_n)p_{\Delta r}(r' - r_m|E_m) \times |K_s(r_n, r_m) - \tilde{g}_d \delta(r_n - r_m)|d^2r_nd^2r_m \times p(E_n)p(E_m)dE_ndE_m. \] (A25)

By assuming WSS conditions \( (K(r, r') = K(r - r')) \) and defining new variables \( \tau = r - r' \), \( \xi = r - r_n \) and \( \xi' = r' - r_m \), Eq. (A25) simplifies to
\[ K_C(\tau, \xi) = \tilde{g}_d \langle f \tilde{g}_C^2 + f \sigma^2_{\tilde{g}_C} \rangle_s \delta(\tau) + \int_A \int_A \langle f \tilde{g}_C p_{\Delta r}(\xi) \rangle_s \langle f \tilde{g}_C p_{\Delta r}(\xi') \rangle_s \times |K_s(\tau - (\xi - \xi')) - \tilde{g}_d \delta(\tau - (\xi - \xi'))|d^2\xi d^2\xi'. \] (A26)
Finally, letting $\ast$ and $\star$ denote convolution and correlation operators, respectively, the autocovariance can be rewritten as

$$K_c(\tau) = \tilde{q}_s(\tau f_{yd}^2 + f_{yd}) \delta(\tau) + [K_s(\tau) - \tilde{q}_s(\delta(\tau))] \ast [f_{yd} P_{\Delta r}(\tau)] \ast (f_{yd} P_{\Delta r}(\tau))_s.$$  \hspace{1cm} (A27)

Taking the Fourier transform gives the Wiener NPS:

$$W_C(k) = |W_s(k) - \tilde{q}_s|\cdot 2 q_{rs} + \tilde{q}_s(2 q_{rs} + \sigma_{rs}^2)_s$$  \hspace{1cm} (A28)

where $T(k)$ is the Fourier transform of $P_{\Delta r}(\tau)$.

Similar to the result for path C, the Wiener NPS of path B is given by

$$W_B(k) = |W_s(k) - \tilde{q}_s|\cdot 2 q_{rs} + \tilde{q}_s(2 q_{rs} + \sigma_{rs}^2)_s.$$  \hspace{1cm} (A29)

Previous work from our group has shown that the parallel branch “Cascade fork” in Fig. 14 has a nonzero cross term equal to the mean product of all gains and scatter transfer functions:

$$R_{BC}(r,r') = E[\tilde{q}_C(\tau) q_C(r')]$$

$$= E \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{q}_{BN}(\tilde{E}_m \tilde{E}_n) \tilde{q}_C(\tilde{E}_m) \right\}$$

and $K_{BC}(r,r') = R_{BC}(r,r') - \tilde{q}_s q_C(r')$. Similar to the derivation above, the crosscovariance for WSS conditions is

$$K_{BC}(r) = \tilde{q}_s(\tilde{f}_{yd} P_{\Delta r}(\tau))_s$$

and the NPS cross term is therefore given by

$$W_{BC}(k) = |W_s(k) - \tilde{q}_s|\cdot 2 q_{rs} + \tilde{q}_s(2 q_{rs} + \sigma_{rs}^2)_s.$$  \hspace{1cm} (A32)

The summation of Eqs. (A28), (A29), and (A32) is a general result for the NPS at the output of the energy-labeled scatter reabsorption process shown in Fig. 14. For the special case of Poisson-distributed and uncorrelated $q_s$, where $W_s = \tilde{q}_s$ as used in this model, the sum reduces to

$$W_s(k) = \tilde{q}_s(2 q_{rs} + \sigma_{rs}^2)_s + 2(\tilde{f}_{yd} P_{\Delta r}(\tau))_s$$  \hspace{1cm} (A33)

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