Applied Fourier Transforms

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Outline

• $h(x) \leftrightarrow H(u)$
• MTF
• Band-limited signal
• Nyquist criterion
• Aliasing
Optical transfer function (OTF)
- Fourier transform of PSF or LSF
- \( \text{OTF}(u) = \mathcal{F}\{h(x)\} = \mathcal{F}\{\text{lsf}(x)\} \)
  - Relative amplitude & phase shift of a sinusoidal target as a function of spatial frequency
  - Spatial frequency: the distinguishable number of line pairs per millimeter (lp/mm)

Modulation transfer function (MTF)
- Amplitude of OTF
- \( \text{MTF}(u) = |\text{OTF}(u)| \)
- Spatial resolution = distinguishable number of line pairs per millimeter (lp/mm)
- Alternatively, the spatial frequency at a specified small amplitude of MTF (e.g., 10%)
- Phase transfer function (PTF): the phase component of OTF
Figure 3. Conceptual illustration of the MTF. The inputs are sine waves with unit amplitude and different frequencies. At the output of the system, the frequency of each sine wave is unchanged, but the amplitudes are decreased. For each sine wave, the peak amplitude is decreased by the same amount, so that the output amplitude is constant for a given sine wave. The amount of degradation of the amplitude is dependent on the frequency of the sine wave. A plot of the output amplitude as a function of the frequency of the sine wave gives an MTF curve. If the input amplitude of each sine wave is not 1.0, then the MTF is determined by calculating the ratio of the output amplitude to the input amplitude.
Polar form

Using polar coordinates: \( x = r \cos \theta \)
\( y = r \sin \theta \)

Forward Fourier transform:

\[
S(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(x, y) e^{-i2\pi(ux+vy)} \, dx \, dy
\]
\[
= \int_{0}^{2\pi} \int_{0}^{\infty} s(r, \theta) e^{-i2\pi(ur \cos \theta+vr \sin \theta)} \, r \, dr \, d\theta
\]
\[
= \int_{0}^{\pi} \int_{-\infty}^{\infty} s(r, \theta) e^{-i2\pi(ur \cos \theta+vr \sin \theta)} \, |r| \, dr \, d\theta
\]

Note the Jacobian of the transformation

\[
J \equiv \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r
\]

Inverse Fourier transform:

\[
s(x, y) = \int_{0}^{\pi} \int_{-\infty}^{\infty} S(f, \phi) e^{i2\pi(xf \cos \phi+yr \sin \phi)} \, |f| \, df \, d\phi
\]

where \( f = \sqrt{u^2 + v^2} \) & \( \tan^{-1} \left( \frac{v}{u} \right) \)
Sampling

Continuous signal: \( s(x), \forall x \in \mathbb{R} \)

Sampled signal: \( s_s(x) = s(n\Delta x), n \in \mathbb{Z} \)

where \( \Delta x = \) sampling distance

How to recover \( s(x) \) completely from its samples \( s_s(x) \)?

- **Sampling theorem or Nyquist criterion**
  - If the Fourier transform of a given signal is *band limited* & if the sampling frequency is *larger than twice the maximum spatial frequency* present in the signal, then the samples uniquely define the given signal

\[
S(u) = 0 \text{ for } \forall |u| > u_{\text{max}}
\]

\[
\frac{1}{\Delta x} = u_s > 2u_{\text{max}}
\]

then \( s_s(x) = s(n\Delta x) \) uniquely defines \( s(x) \)
Comb function or impulse train: \[\mathbb{W}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)\]

Then, the sampled signal: \[s_s(x) = s(x) \cdot \mathbb{W}(x)\]

Applying the convolution theorem:

\[S_s(u) = S(u) \ast \mathcal{F}\{\mathbb{W}(x)\}\]

\[= S(u) \ast u_s \sum_{l=-\infty}^{\infty} \delta(u - lu_s)\]

\[= u_s[S(u) + S(u - u_s) + S(u + u_s) + S(u - 2u_s) + S(u + 2u_s) + \ldots]\]

Note that \(S(u) = 0\) for \(\forall u \geq \frac{u_s}{2}\)

or equivalently \(S(u) = S_s(u)\frac{1}{u_s} \prod \left(\frac{u}{u_s}\right)\)

Consequently \(s(x)\) can be recovered from \(S_s(u)\)
signal w/ an *infinite* spatial extent

band-limited Fourier transform

comb function

\[ s_s(x) = s(x) \cdot \Pi(x) \]

\[ S_s(u) = S(u) \ast \mathcal{F}\{\Pi(x)\} \]
Finite spatial extent

- If \( s(x) \) is not band limited, or if it is band limited but \( \frac{1}{\Delta x} (= u_s) \leq 2u_{\text{max}} \), the shifted replicas of \( S(u) \) (called the *aliases*) will overlap.

- The spectrum of \( s(x) \) cannot be recovered by multiplication with a rectangular pulse.

- This phenomenon is known as *aliasing* & is unavoidable if the original signal \( s(x) \) is not band limited.

- Note that a patient *always* has a *limited spatial extent*, which implies that the FT of an image of the body is *never* band limited.
signal w/ an \textit{finite} spatial extent

\[ s(x) = s(x) \cdot \mathcal{M}(x) \]

not band-limited

\[ S_s(u) = S(u) \ast \mathcal{F}\{\mathcal{M}(x)\} \]
SPATIAL DOMAIN

a) $d(x)$

b) $\times \sum_n \delta(x - nx_s)$

c) $d^\dagger(x) = \sum_n d_n \delta(x - nx_s)$

SPATIAL-FREQUENCY DOMAIN
(Magnitude)

$D(u)$

$\ast u_s \sum_n \delta(u - nu_s)$

$= u_s [D(u) + D(u \pm u_s) + \ldots]$
Anti-aliasing

star pattern test image:
spatial frequencies increase towards the center

original -> subsample -> aliasing artifacts
blur (lowpass) -> aliasing artifacts still exist, just blurred
....so the answer is ‘No’

only little aliasing

subsample
Discrete FT

• Forward Fourier transform

\[ S(m\Delta u, n\Delta v) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} s(p\Delta x, q\Delta y) e^{-2\pi i \left( \frac{mp}{M} + \frac{nq}{N} \right)} \]

• Inverse Fourier transform

\[ s(p\Delta x, q\Delta y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} S(m\Delta u, n\Delta v) e^{2\pi i \left( \frac{mp}{M} + \frac{nq}{N} \right)} \]

• Fast Fourier transform (FFT)
  – Number of samples in a power of 2 \((2^n)\)
  – \(n \log n\) flops in 1D
  – \(n^2 \log n\) flops in 2D
Wrap-up

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- MTF
- Band-limited signal
- Nyquist criterion
- Aliasing